Physics of complex systems and criticality

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$T \gg T_c$

critical opalescence

$T \approx T_c$

$T \ll T_c$
What is a critical state?

- **Ising model**: two-state spins
- **Ferromagnetic interaction**: the energy is lower if spins are parallel
- **System evolves toward equilibrium state which minimizes the energy**
- **New state of a spin is assigned according to the probability**

\[ \frac{E}{K_B T} \propto e \]

- At \( T \approx \infty \) all spins are uncorrelated
- At \( T = 0 \) all spins tend to align

**Emergence of spontaneous magnetization at** \( T_c \) (\( h=0 \))
For $h=0$ Hamiltonian of the Ising model is symmetric for spin inversion.

Configurations with magnetization $+m$ and $-m$ have the same probability

$\langle m \rangle = 0$  \hspace{1cm} no spontaneous magnetization!

In the limit $N \rightarrow \infty$ the symmetry between the two phases can be broken and a spontaneous magnetization emerges.

The system is trapped in a phase space subregion since the energy barrier is too high.

Probability to escape $p \propto e^{-N^{(d-1)/d}}$.
$T \approx 0$

$T \geq T_c$

$T \approx T_c$

$T \rightarrow \infty$
clusters of all possible sizes are present
divergence of correlation range
divergence of fluctuations
Self-similarity the largest cluster is fractal

At the critical point
physical properties behave as power laws:
Order parameter
Response function
Specific heat
.......
Scale invariance - Mandelbrot set
SELF-SIMILARITY

Salvador Dalí “The war's face”

A part looks like the whole....
Self-similarity

Diffusion Limited Aggregation
Correlation function

The correlation function

\[ G(r) = \langle m(r)m(0) \rangle - \langle m(0) \rangle^2 \]

measures the correlations between fluctuations in 0 and r

The spatial extent of the correlations is called correlation length \( \xi \)

The spatial integral of the correlation function provides the response function (derived by the fluctuation dissipation theorem):

\[ \chi = \frac{1}{k_B T} \int d^d r G(r) \]

At the critical point

\[ \xi \rightarrow \infty \]

Experimentally can be measured by scattering experiments

Structure factor

\[ S(q = 0) \propto \tilde{G}(0) = \int G(\tilde{r}) d\tilde{r} \]
Critical Exponents

Reduced Temperature, $t$

$$t \equiv \frac{T - T_c}{T_c}$$

Specific heat $C \propto |t|^{-\alpha}$

Magnetization $M \propto |t|^{\beta}$

Magnetic susceptibility $\chi \propto |t|^{-\gamma}$

Correlation length $\xi \propto |t|^{-\nu}$

The exponents display critical point universality (don’t depend on details of the model).

Universality classes!
Densities of coexisting liquid and gas phases of a variety of substances, plotted against temperature, with both densities and temperatures scaled to their value at the critical point.

Exponents are not all independent....

\[ \alpha + 2\beta + \gamma = 2, \]
\[ \gamma = \beta(\delta - 1), \]
\[ 2 - \alpha = \nu d, \]
\[ \frac{\gamma}{\nu} = 2 - \eta. \]

A number of relations have been derived among them

**PROBLEM:** thermodynamics predicts them as **INEQUALITIES**

Whereas experimental values satisfy **EQUALITIES!!!**
Scaling - Widom hypothesis

- Near the critical point the main physical properties exhibit power law behaviour
- Nice properties of power laws invariant under rescaling!

Suppose \( y = f(x) = x^\alpha \)

Make the scale transformation

\[ x \rightarrow x' = bx \quad y \rightarrow y' = cy \]

Under rescaling

\[ f(x) \rightarrow f(x') = cf(x' / b) \]

If \( c = b^\alpha \) the function is invariant

\[ f(\lambda x) = g(\lambda) f(x) \]

Scale invariance homogeneous functions
Power law

compress $\Rightarrow y'/c = f(x'/b)$ \hspace{1cm} c=b\(^2\)

stretch $\Rightarrow y'=cy=f(x'=bx)$ \hspace{1cm} b=2

$y=f(x)=x^2$
Exponential function

- compress $\rightarrow y'/c = f(x'/b)$  $c = e^b$
- stretch $\rightarrow y'=cy = f(x'=bx)$  $b = 2$
- $y = f(x) = e^x$
For functions of more than one variable

\[ f(\lambda^a x, \lambda^b y) = \lambda f(x, y) \quad \forall \lambda \]

Choose \[ \lambda = y^{-1/b} \]

Obtain

\[ f(x, y) = y^{1/b} f(y^{-a/b} x, 1) = y^{1/b} g(y^{-a/b} x) \]

Where \( g \) is an universal function

Assuming that thermodynamic potentials are GHF leads to relations among critical exponents as equalities
What is a complex system?

- Many components or degrees of freedom
- Interactions among components ⟷ cooperative effects
- Emergence of «impredictable» macroscopic behaviour

What is the signature of a complex system?

- Fundamental properties exhibit singular behaviour
- Emergence of power laws
- Absence of a characteristic scale
“I think you should be more explicit here in step two.”
Complex systems
A single block pulled on a rough surface slips always the same distance:

\[ \Delta x = (\mu_s - \mu_d)(N/K) \]

On real faults earthquakes of all size are measured.
From complexity

Bull by Picasso

To Universality
Power laws in nature


Forest fires in Ontario (Canada) 1976-1996 Turcotte & Malamud 2004

Areas covered by lava in volcanic eruptions (Springerville, Arizona) Lahaie & Grasso 1998
SELF-ORGANIZED CRITICALITY

Dynamical systems spontaneously evolving toward a critical state without parameter tuning

no characteristic event size

Sand pile

by adding at random one grain...

threshold=4

Size and duration distribution

P(s) \sim s^{-1}

P(T) \sim T^{-0.5}
Fundamental ingredient: separation of time scales
- Slow scale: adding a grain
- Fast scale: propagation of an avalanche

SOC applied to many natural phenomena
- Slides and avalanches
- Neural activity
- Solar flares
- Fluctuations in confined plasma
- Biological evolution
- Earthquakes
The process generated by the sandpile or other standard SOC models is Poissonian → absence of temporal correlations

Additional ingredients must be introduced to generate a correlated process

In many stochastic processes in nature temporal correlations are present

How can we detect them?
Intertime distribution

Probability distribution of intertimes $\Delta t$
between consecutive events

- $P(\Delta t)$ is an exponential for a Poisson process

- It exhibits a more complex structure as temporal correlations are present in the process

Barabasi, Nature 2005
Corral (PRL, 2004) rescaling $\Delta t$ by the average rate in the area obtained a universal scaling law for the probability density

$$D(\Delta t, M_c) = R(M_c) f(R(M_c)\Delta t)$$

holds also for Japan, Spain, New Zealand...
scaling function not universal
(different areas are characterized by different rates)
Wiener – Khintchine Theorem

Autocorrelation function

\[ K(s) = K(t_2 - t_1) = \langle A(t_2)A(t_1) \rangle \]

Power spectrum

\[ S(f) = 4 \int_0^\infty dsK(s) \cos(2\pi fs) \]
If the variable is very irregular (unpredictable) Then $K(s) = c \delta(s)$ and $S(f) = 2c$ for all $f$

--- White noise ---

but $K(0)$ would diverge!

In reality $K(s)$ decays sharply within $\tau$

Then $S(f)$ is constant over a frequency range $1/\tau$
Color of noise

When not white noise is colored

- power law behaviour \( f^{-\beta} \)
- Pink (flicker) \( \beta = 1 \)
- Brown (red) noise \( \beta = 2 \)
  (by integrating white noise)

\( f^{-1} \) noise \( \longrightarrow \) long range temporal correlations

\( f^{-2} \) noise \( \longrightarrow \) uncorrelated signal
Take-home message

- Criticality implies the absence of a characteristic scale
- Emergence of power laws can be explained by SOC
- Power law distributions are not sufficient for criticality
- Necessity to verify the existence of long-range temporal correlations